MSc. (1^{st} sem) Exam 2015-16
(Mathematical Analysis)

1.(i) $m^*\{1,2,3,4\} = 0$ $\{\because$ Lebesgue outer measure of countable set is $0\}$

(ii) Let $P = \{x_0 = a, x_1, x_2 \dots x_n = b\}$ is any partition of $[a,b]$

then $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ where $M_i = \sup_{x\in[x_{i-1},x_i]} f(x) = 1$

$= \sum_{i=1}^{n} \Delta\alpha_i = \Delta\alpha_1 + \Delta\alpha_2 + \dots + \Delta\alpha_n$

~~$= (\alpha_1-\alpha_0)+(\alpha_2-\alpha_1)+\dots+\alpha_n-\alpha_{n-1}$~~

$= (\alpha(x_1)-\alpha(x_0)) + (\alpha(x_2)-\alpha(x_1)) + \dots + \alpha(x_n)-\alpha(x_{n-1})$

$= \alpha(x_n)-\alpha(x_0) = x_n - x_0 = b-a$

$\therefore \overline{\int_a^b} f d\alpha = \inf U(P,f,\alpha) = b-a$

(iii) Lebesgue Differentiation Theorem.- Let f be an increasing real-valued function defined on $[a,b]$. Then f is differentiable a.e. and the derivative f' is measurable. Furthermore,

$$\int_a^b f'(x)dx \le f(b)-f(a).$$

(iv) A non-negative extended real-valued set function $\mu^*: P(X) \to [0,\infty]$ defined on the power set $P(X)$ of some set X is called an Caratheodory outer measure if it satisfies the following properties:

1) $\mu^*(\phi) = 0$

2) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$, i.e. μ^* is monotone.

3) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ holds for every sequence $\{E_n\}$ of subsets of X; that is, μ^* is σ-additive.

(V) Convex function :- A real-valued function $\phi : (a,b) \to R$ defined on a segment (a,b) where $-\infty \leq a < b \leq \infty$, is said to be convex if for each $x, y \in (a,b)$ and each λ, $0 \leq \lambda \leq 1$, the inequality

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \text{ holds.}$$

(VI)
$$D^+ f(0) = \overline{\lim_{h \to 0^+}} \frac{f(h) - f(0)}{h} = \overline{\lim_{h \to 0^+}} \left(\sin\frac{1}{h}\right)$$

$$= \inf_{\delta > 0} \sup\left\{\sin\frac{1}{h} : 0 < h < \delta\right\} = \inf_{\delta > 0} \{1\} = 1$$

$$D_+ f(0) = \underline{\lim}_{h \to 0^+} \left(\sin\frac{1}{h}\right) = \sup_{\delta > 0} \inf\left\{\sin\frac{1}{h} : 0 < h < \delta\right\}$$

$$= -1$$

$$D^- f(0) = \overline{\lim_{h\to 0^-}} \frac{f(h)-f(0)}{h} = \overline{\lim_{h\to 0^-}} \frac{h\sin 1/h + 4h}{h}$$

$$= \overline{\lim_{h\to 0^-}} \left(\sin 1/h + 4\right) = \inf_{\delta>0} \sup\left\{\sin\frac{1}{h}+4\right\}$$

$$= 5$$

Similarly $D_- f(0) = 3$

(vii) $\int_E f \, d\mu = 0$ if $\mu(E) = 0$

(viii) Minkowski's Inequality. Let $1 \le p \le \infty$ & let $f, g \in L^p(\mu)$. Then $f+g \in L^p(\mu)$ and

$$\|f+g\|_p \le \|f\|_p + \|g\|_p .$$

(ix) Ex 1. $f: R \to R$ defined by $f(x) = x^2$

Ex 2. $g: R \to \{0,1\}$ defined by

$$g(x) = \begin{cases} 1 & x \in [0,3] \\ 0 & \text{otherwise} \end{cases}$$

(x) $\because m^*$ is translation invariant

i.e. $m^*(A+x) = m^*(A)$ for every set A & for $x \in R$

taking $x = 3$, we get $m^*(A+3) = m^*(A) = 8$ Ans.

2. <u>The if part.</u> Assume that for every $\varepsilon > 0$ $\exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \quad \cdots (1)$$

For every partition P of $[a,b]$, we have

$$L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) \quad \cdots (2)$$

From (1) & (2), we conclude that

$$0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Thus $\quad 0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon$

Since ε is arbitrary, we have

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \Rightarrow f \in R(\alpha)$$

<u>The only if Part.</u> Suppose $f \in R(\alpha)$. Then

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \quad \text{——— } (3)$$

Let $\varepsilon > 0$ be given. Since $\underline{\int_a^b} f\,d\alpha$ is the supremum of $L(P,f,\alpha)$ over all partition $P \in P[a,b]$, $\exists$ a partition $P_1 \in P[a,b]$ such that

$$\underline{\int_a^b} f\,d\alpha < L(P_1,f,\alpha) + \frac{\varepsilon}{2} \quad \text{——— } (4)$$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - L(P_1,f,\alpha) + \varepsilon/2 \quad \text{——— } (4)$$

Similarly $\exists$ $P_2 \in P[a,b]$ Such that

$$U(P_2, f, \alpha) < \overline{\int_a^b} f\, d\alpha + \frac{\varepsilon}{2}. \quad \text{--- (5)}$$

Let $P = P_1 \cup P_2$

then from (4) & (5)

$$\underline{\int_a^b} f\, d\alpha - L(P, f, \alpha) < \varepsilon/2 \quad \text{--- (6)}$$

$$\& \quad U(P, f, \alpha) < \overline{\int_a^b} f\, d\alpha + \varepsilon/2 \quad \text{--- (7)}$$

Adding (6) & (7), we get

$$\underline{\int_a^b} f\, d\alpha + U(P, f, \alpha) - L(P, f, \alpha) < \overline{\int_a^b} f\, d\alpha + \varepsilon$$

using (3), this reduces to

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

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3. Since γ' is continuous on $[a,b]$, hence it is bounded $\Rightarrow$ γ is of bounded variation on $[a,b]$ and is therefore rectifiable on $[a,b]$.

If $a \le x_{i-1} < x_i \le b$, then by property of Riemann integrable function, we have

$$\left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

[$\because$ γ' is continuous on $[a,b]$ & γ' is Riemann integrable in any sub-interval of $[a,b]$]

But $\left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| = \left| \left[\gamma(t) \right]_{x_{i-1}}^{x_i} \right| = \left| \gamma(x_i) - \gamma(x_{i-1}) \right|$

Thus $\left| \gamma(x_i) - \gamma(x_{i-1}) \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt$

$$\Rightarrow \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt$$

$$\Rightarrow \quad \Lambda_\gamma(P) \leq \int_a^b |\gamma'(t)|\,dt \quad \text{for every partition } P \in \mathcal{P}[a,b]$$

$$\therefore \sup_{P \in \mathcal{P}[a,b]} \Lambda_\gamma(P) \leq \int_a^b |\gamma'(t)|\,dt \Rightarrow \Lambda_\gamma(a,b) \leq \int_a^b |\gamma'(t)|\,dt \quad \text{——— (1)}$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since every continuous mapping defined on a compact set is uniformly continuous, it follows that γ' is uniformly continuous on $[a,b]$. Thus, any given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|s-t| < \delta \Rightarrow |\gamma'(s) - \gamma'(t)| < \varepsilon \quad \text{——— (2)}$$

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$, with $\Delta x_i < \delta \quad \forall\, i = 1, 2, \ldots, n$.

If $x_{i-1} \leq t \leq x_i$ then

$$\begin{aligned} |\gamma'(t)| &= |\gamma'(t) - \gamma'(x_i) + \gamma'(x_i)| \\ &\leq |\gamma'(t) - \gamma'(x_i)| + |\gamma'(x_i)|, \text{ as } |t - x_i| \leq \Delta x_i < \delta \\ &< |\gamma'(x_i)| + \varepsilon \end{aligned}$$

$$\therefore \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt < (|\gamma'(x_i)| + \varepsilon) \int_{x_{i-1}}^{x_i} dt$$

$$|\gamma'(x_i)|\,\Delta x_i + \varepsilon \Delta x_i = \left|\int_{x_{i-1}}^{x_i}\left[\gamma'(t)+\gamma'(x_i)-\gamma'(t)\right]dt\right| + \varepsilon\Delta x_i$$

$$\leq \left|\int_{x_{i-1}}^{x_i}\gamma'(t)\right| + \left|\int_{x_{i-1}}^{x_i}\left[\gamma'(x_i)-\gamma'(t)\right]dt\right| + \varepsilon\Delta x_i$$

$$< |\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon\Delta x_i \qquad \forall i=1,2,\dots,n$$

Adding all these inequalities, we obtain

$$\int_a^b |\gamma'(t)|\,dt < \sum_{i=1}^{n}|\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon\sum_{i=1}^{n}\Delta x_i$$

i.e., $\displaystyle\int_a^b |\gamma'(t)|\,dt < \Lambda_\gamma(P) + 2\varepsilon(b-a)$

$$\leq \sup \Lambda_\gamma(P) + 2\varepsilon(b-a)$$

$$= \Lambda_\gamma(a,b) + 2\varepsilon(b-a).$$

Letting $\varepsilon \to 0$

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda_\gamma[a,b) \quad \text{——— (3)}$$

From (1) & (3) $\displaystyle\int_a^b |\gamma'(t)|\,dt = \Lambda_\gamma[a,b]$

4. (a) Let a be any real number. Then

$$E(f+g>a) = \{x \in E : f(x)+g(x) > a\}$$
$$= \{x \in E : f(x) > a - g(x)\}$$

Now g is measurable $\Rightarrow$ cg is measurable $\forall a, c \in R$

$\Rightarrow$ $a + cg$ is measurable $\forall a, c \in R$

$\Rightarrow$ $a - g$ is measurable

Hence $\{x \in E : f(x) > a - g(x)\}$ is measurable since $f(x)$ & $a - g(x)$ both are measurable.

$\Rightarrow$ $E(f+g>a)$ is measurable $\forall a \in R$

$\Rightarrow$ $f+g$ is measurable.

(b) f and g are measurable on E $\Rightarrow$ f & $(-1)g$ are measurable on E $\Rightarrow$ f and $-g$ are measurable on E

$\Rightarrow$ $f+(-g)$ i.e. $f-g$ is measurable on E

(c) For $a \geq 0$, we have $E(f^2 < a) = E(|f| < \sqrt{a})$

But $E(|f| < \sqrt{a}) = E(f < \sqrt{a}) \cap E(f > -\sqrt{a})$

$\Rightarrow$ $E(f^2 < a) = E(f < \sqrt{a}) \cap E(f > -\sqrt{a})$ if $a \geq 0$

~~Hence~~ $\because$ $E(f < \sqrt{a})$ & $E(f > -\sqrt{a})$ are measurable

$\therefore$ there intersection $E(f^2 < a)$ is also measurable

For $a < 0$ $\quad E(f^2 < a) = \phi$ = measurable set

$\therefore$ f^2 is measurable on E.

(d) To prove that fg is measurable, we see that f and g are measurable functions

$\Rightarrow f+g$ & $f-g$ are measurable functions

$\Rightarrow (f+g)^2$ & $(f-g)^2$ both are measurable functions

$\Rightarrow (f+g)^2 - (f-g)^2$ is a measurable function

$\Rightarrow \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ is a measurable function

$\Rightarrow fg$ is a measurable function.

(e) f & g are measurable $\Rightarrow f$ & $\frac{1}{g}$ are measurable

$\Rightarrow f \cdot \frac{1}{g}$ i.e $\frac{f}{g}$ is a measurable function.

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5 (a) Lebesgue Dominated Convergence Theorem –

Let g be an integrable function on E & let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and $\lim_{n\to\infty} f_n = f$ a.e., on E.

Then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

Proof : $|f_n| \leq g$ on $E \Rightarrow f_n$ is integrable over E $\forall n \in N$

Further, it follows from $\lim_{n\to\infty} f_n = f$ a.e. on E & $|f_n| \leq g$ on E that $|f| \leq g$ a.e. on E. Hence f is integrable over E.

Setting $h_n = f_n + g$ for each n. Clearly h_n is a non-negative and integrable function for each n. Therefore, by Fatou's Lemma, we get

$$\int_E (f+g) \le \underline{\lim}_{n\to\infty} \int_E (f_n + g).$$

$$\Rightarrow \int_E f + \int_E g \le \underline{\lim}_{n\to\infty} \int_E f_n + \int_E g$$

$$\Rightarrow \int_E f \le \underline{\lim}_{n\to\infty} \int_E f_n \quad \left[\because \int_E g \text{ is finite}\right] \quad \text{--- (1)}$$

Similarly, Consider a sequence $\{k_n\}$ of functions defined by $k_n = g - f_n$, we see that k_n is a non-negative and integrable function for each n. Hence, again by Fatou's Lemma, we have

$$\int_E (g-f) \le \underline{\lim}_{n\to\infty} \int_E (g-f_n) \Rightarrow \int_E g - \int_E f \le \int_E g + \underline{\text{Lt}}_{n\to\infty} - \int_E f_n$$

$$\Rightarrow \int_E f \ge \overline{\lim}_{n\to\infty} \int_E f_n \quad \text{———} \quad (2)$$

Hence from (1) & (2), we obtain

$$\overline{\lim}_{n\to\infty} \int_E f_n \le \int_E f \le \underline{\lim}_{n\to\infty} \int_E f_n$$

$$\Rightarrow \int_E f = \underline{\lim}_{n\to\infty} \int_E f_n = \overline{\lim}_{n\to\infty} \int_E f_n \Rightarrow \int_E f = \lim_{n\to\infty} \int_E f_n.$$

(b) Suppose f is integrable over E

$\Rightarrow$ i.e $f = f^+ - f^-$ is integrable over E

$$\Rightarrow \int_E f^+ - \int_E f^- < \infty \Rightarrow \int_E f^+ < \infty \ \& \int_E f^- < \infty$$

$\Rightarrow f^+ \ \& \ f^-$ is integrable

$$\Rightarrow \int_E (f^+ + f^-) < \infty \Rightarrow \int_E |f| < \infty$$

$\Rightarrow |f|$ is integrable over E.

Conversely, suppose that $|f|$ is Lebegue integrable over E, then $\int_E |f| < \infty$.

Since $0 \leq f^+(x) \leq |f(x)| \ \forall x \in E$

$$\Rightarrow \int_E f^+ \leq \int_E |f| < \infty \Rightarrow \int_E f^+ < \infty$$

$\Rightarrow f^+$ is integrable

Similarly, we can prove that f^- is Lebesgue integrable

But $f = f^+ - f^-$. Now

$$\int_E f^+ < \infty, \ \int_E f^- < \infty \Rightarrow \int_E f^+ - \int_E f^- < \infty \Rightarrow \int_E f < \infty$$

$\Rightarrow f$ is integrable over E.

6. Let x_0 be any point in $[a,b]$. Then

$$|F(x)-F(x_0)| = \left|\int_a^x f(t)\,dt + c - \int_a^{x_0} f(t)\,dt - c\right|$$

$$= \left|\int_{x_0}^{x} f(t)\,dt\right| \leq \left|\int_{x_0}^{x} |f(t)|\,dt\right|$$

But f is integrable over $[a,b]$

$\Rightarrow$ $|f|$ is integrable over $[a,b]$

$\therefore$ given $\varepsilon>0$ $\exists\ \delta>0$ such that for every measurable set $A\subset[a,b]$ with $m(A)<\delta$, we have

$$\int_A |f| < \varepsilon.$$

In particular, for $|x-x_0|<\delta$ we have

$$\left|\int_{x_0}^{x} |f(t)|\,dt\right| < \varepsilon.$$

Hence $|F(x)-F(x_0)|<\varepsilon$, whenever $|x-x_0|<\delta$ —

$\Rightarrow$ F is continuous at x_0 & hence in $[a,b]$.

To show that F is a function of bounded variation, let

$P=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be a partition of $[a,b]$. Then

$$\sum_{i=1}^{n}|F(x_i)-F(x_{i-1})| = \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} f(t)\,dt\right| \leq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}|f(t)|\,dt$$

$$= \int_a^b |f(t)|\,dt$$

i.e. $V_a^b(F,P) \leq \int_a^b |f(t)|\,dt \Rightarrow T_a^b(F) = \sup_{P\in\mathcal{P}[a,b]} V_a^b(F,P) \leq \int_a^b |f(t)|\,dt$

$\Rightarrow T_a^b(F) < \infty \quad \left[\because \int_a^b |f(t)|\,dt < \infty\right]$

7. First to prove that $L^p(\mu)$ is a linear space, we have to show that

(i) $f, g \in L^p(\mu) \Rightarrow f+g \in L^p(\mu)$ and

(ii) $c \in R,\ f \in L^p(\mu) \Rightarrow cf \in L^p(\mu)$.

To prove (i), we see that

$$|f+g|^p \le \big|\,|f| + |g|\,\big|^p \le \left[2\max\{|f|, |g|\}\right]^p = 2^p \max\{|f|^p, |g|^p\}$$
$$\le 2^p\left(|f|^p + |g|^p\right)$$

$$\therefore \int_X |f+g|^p\,d\mu < 2^p \int_X \left(|f|^p + |g|^p\right) d\mu$$
$$= 2^p\left(\int_X |f|^p\,d\mu + \int_X |g|^p\,d\mu\right) < \infty.$$

$\Rightarrow f+g \in L^p(\mu)$

To prove (ii), we see that for any $a \in R$ and $f \in L^p(\mu)$,

$$\int_X |af|^p\,d\mu = |a|^p \int_X |f|^p\,d\mu < \infty.$$

Thus $a \in R,\ f \in L^p(\mu) \Rightarrow af \in L^p(\mu)$.

We now show that $L^p(\mu)$ is a normed linear space.

(N_1) $|f| \ge 0 \Rightarrow \left(\int_X |f|^p\,d\mu\right)^{1/p} \ge 0 \Rightarrow \|f\|_p \ge 0$

(N_2) By Minkowski's inequality for $1 \le p < \infty$, we have

$$\|f+g\| \le \|f\|_p + \|g\|_p \quad \forall\ f, g \in L^p(\mu).$$

(N3) Let $a \in \mathbb{R}$ & $f \in L^p(\mu)$ be arbitrary. Thus

$$\|af\|_p = \left(\int_X |af|^p d\mu\right)^{1/p} = \left(|a|^p \int_X |f|^p d\mu\right)^{1/p}$$

$$= |a| \left(\int_X |f|^p d\mu\right)^{1/p} = |a| \, \|f\|_p .$$

(N4) If $\|f\|_p = 0$, then it follows that $f = 0$ a.e. Now ~~In other words~~, identity functions which are equal a.e.

then $\|f\|_p = 0 \Leftrightarrow f = 0$

Thus $L^p(\mu)$ is a normed linear space.

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8 (a) Let $\varepsilon > 0$ be given. Then we observe that

$$\{x : |f_n(x) - g(x)| \geq \varepsilon\} \subseteq \{x : f(x) \neq g(x)\} \cup \{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

$$\Rightarrow \mu^*\left(\{x : |f_n(x) - g(x)| \geq \varepsilon\}\right) \leq \mu^*\left(\{x : f(x) \neq g(x)\}\right) + \mu^*\left(\{x : |f_n(x) - f(x)| \geq \varepsilon\}\right)$$

$$\Rightarrow \mu^*\{x : |f_n(x) - g(x)| \geq \varepsilon\} \leq \mu^*\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

since $\mu^*(\{x : f(x) \neq g(x)\}) = 0$

This proves the result.

8(b) Let $g \in M$ = set of measurable functions, be distinct from f such that $f_n \xrightarrow{\mu} g$.

Now $\because |f-g| \leq |f-f_n| + |f_n - g|$.

$\therefore$ for each $\epsilon > 0$, we have

$$\{x : |f(x) - g(x)| \geq \epsilon\} \subset \{x : |f(x) - f_n(x)| \geq \tfrac{\epsilon}{2}\} \cup \{x : |f_n(x) - g(x)| \geq \tfrac{\epsilon}{2}\}$$

By a proper choice of ϵ, the measure of both the sets on the right can be made as small as we please, so we have $\mu^*(\{x : |f(x) - g(x)| \geq \epsilon\}) = 0$

$\Rightarrow f \sim g$.